NEIGHBORHOODS OF EXTREME POINTS(1)

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ABSTRACT

An examination of relationship between two neighborhood systems (relative to two linear topologies) of extreme points yields a unified approach to some known and new results, among which are Bessaga-Pełczyński's theorem on closed bounded convex subsets of separable conjugate Banach spaces and Ryll-Nardzewski's fixed point theorem.

§0. **Introduction.** Let C be a compact subset of a Banach space E. Then, of course, the norm topology and the weak topology agree on C. Now suppose that C is only weakly compact. Then the identity map: $(C, \text{weak}) \to (C, \text{norm})$ is no longer continuous in general. Nevertheless one may still ask how the set of points of continuity of this map is distributed in C. In particular, when C is convex as well as weakly compact, is the identity map: $(C, \text{weak}) \to (C, \text{norm})$ continuous at any of the extreme points of C, i.e., do there exist extreme points of C which have weak neighborhoods (relative to C) of arbitrarily small diameter? The importance of an answer to such a question is demonstrated in Rieffel [7] and in note [6]. Professor J. L. Kelley also recognized the relevance of this question to Ryll-Nardzewski's fixed point theorem. The work of Lindenstrauss in [4] yields the following answer: if C is a weakly compact, convex subset of a separable Banach space, then there are "many" extreme points of C, where the identity map $(C, \text{weak}) \to (C, \text{norm})$ is continuous. This fact was proved by using deep Banach space techniques due to Kadec and Lindenstrauss. In the present article, we shall generalize this result in various directions. The main theorem of this paper (Theorem 2.3) is stated in somewhat obscure, if not pedantic, language, because we tried to combine all the generalizations into one theorem. However, we hope this is forgiven because of the diverse applications of the single theorem. Here are some of the consequences of the main theorem: each bounded subset of a separable, conjugate Banach space is "dentable" in the sense of [7]; each closed, convex, bounded subset of E is the closed convex hull of its extreme points, where E is either a separable, conjugate Banach space or a Fréchet space such that E^{**} is separable relative to its strong topology. In addition, a slight generalization of Ryll-Nardzewski's fixed point theorem can easily be derived from the main theorem.

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The paper is organized as follows: Section §0, the present one, is the introduction. Section §1 contains the preliminary material, and section §2 is devoted to the main theorem. Our proof of the main theorem is independent of Lindenstrauss' work and is quite different in spirit. Category plays a large rôle throughout §§1–2. Section §3 gives applications of the main theorem. Our terminology and notation will be those of Kelley, Namioka, et al. [3].

Finally, we wish to thank R. Phelps for many enlightening discussions on the subject of the present paper.

1. **Preliminaries.** Let $(E,\mathscr{T})$ be a linear topological space, and let A be a subset of E. Then we denote by $(A,\mathscr{T})$ the space A with the topology induced by $\mathscr{T}$. If p is a pseudo-norm on the linear space E, then $\mathscr{T}_p$ denotes the pseudo-norm topology on E given by p. The pseudo-norm p is *lower $\mathscr{T}$-semicontinuous* if $\{x : p(x) \leqq 1\}$ is $\mathscr{T}$-closed. If V is a convex, circled, $\mathscr{T}$-closed subset of E, which is radial at 0,(2) then the Minkowski functional of V is lower $\mathscr{T}$-semicontinuous. For instance, if E is a normed linear space, then the norm on E is lower w-semicontinuous, and the norm on the dual E^* is lower w^*-semicontinuous, where w and w^* are the topologies $w(E,E^*)$ and $w(E^*,E)$ respectively.

1.1 Lemma. *Let X be a compact Hausdorff space, and let $\{C_i : i = 1,2,\cdots\}$ be a sequence of closed subsets of X such that $X = \bigcup\{C_i\colon i = 1,2,\cdots\}$. Then $\bigcup\{\operatorname{Int} C_i\colon i = 1,2,\cdots\}$ is dense in X, where $\operatorname{Int} C_i$ is the interior of C_i in X.*

Proof. We may assume that $X \neq \varnothing$. Let U be an open nonempty subset of X. Then U is locally compact, and hence U is of the 2nd category in itself. Since $U = \bigcup\{U \cap C_i\colon i = 1,2,\cdots\}$ and $U \cap C_i$ is closed in U, for at least one i, $U \cap C_i$ has non-empty interior relative to U and hence relative to X. Therefore, $U \cap \bigcup\{\operatorname{Int} C_i\colon i = 1,2,\cdots\} \neq \varnothing$, and since, U is arbitrary, $\bigcup\{\operatorname{Int} C_i\colon i = 1,2,\cdots\}$ is dense in X.

1.2 Proposition. *Let $(E,\mathscr{T})$ be a Hausdorff linear topological space, let p be a lower $\mathscr{T}$-semicontinuous pseudo-norm such that $(E,\mathscr{T}_p)$ is separable, and let K be a $\mathscr{T}$-compact subset of E. The set of all points of continuity of the identity map: $(K,\mathscr{T}) \to (K,\mathscr{T}_p)$ is a dense G_δ subset of $(K,\mathscr{T})$.*

Proof. For a subset X of E, define $p\text{-diam}(X) = \sup\{p(x-y)\colon x,y \in X\}$. For each $\varepsilon > 0$, let A_ε be the union of all open subsets of $(K,\mathscr{T})$ of p-diam $\leqq \varepsilon$. Clearly A_ε is open. Let $S = \{x : p(x) \leqq \varepsilon/2\}$. Then, since $(E,\mathscr{T}_p)$ is separable, there is a sequence $\{x_i\}$ in E such that $K = \bigcup\{K \cap (x_i + S)\colon i = 1,2,\cdots\}$, and each $K \cap (x_i + S)$ is $\mathscr{T}$-closed because p is lower $\mathscr{T}$-semicontinuous. Hence, by Lemma 1.1, the union of the interiors of $K \cap (x_i + S)$ in $(K,\mathscr{T})$ is dense in K, and this union is clearly contained in A_ε. Therefore A_ε is a dense open subset

(2) V is radial at 0, if, for each x in E, there is a positive number t such that $sx \in V$ whenever $0 \leqq s \leqq t$.

of $(K,\mathscr{T})$. Now the set of points of continuity of the identity map: $(K,\mathscr{T}) \to (K\ \mathscr{T}_p)$ is precisely $\bigcap \{A_{1/n}: n = 1,2,\cdots\}$, and this set is dense in $(K,\mathscr{T})$, because $(K,\mathscr{T})$ is of the 2nd category in itself.

In order to state the corollary, we introduce the following notion, which will be useful in the sequel as well. A *locally convex linear bitopological space* is a triple $(E;\ \mathscr{T}_1, \mathscr{T}_2)$ such that $\mathscr{T}_1$ and $\mathscr{T}_2$ are locally convex vector topologies for the linear space E and such that there is a local base for $\mathscr{T}_1$ consisting of $\mathscr{T}_2$-closed sets. Clearly the last condition is equivalent to: a family $\{p_\alpha\}$ of $\mathscr{T}_1$-continuous, lower $\mathscr{T}_2$-semicontinuous pseudo-norms *determines* the topology $\mathscr{T}_1$ i.e. a net $\{x_\gamma\}$ $\mathscr{T}_1$-converges to x if and only if $\lim_\gamma p_\alpha(x_\gamma - x) = 0$ for each α. Let $(E,\mathscr{T})$ be a locally convex linear topological space, and let E^* be its dual. Then $(E;\mathscr{T}, w(E,E^*))$, $(E^*;\ s(E^*,E), w(E^*,E))$ and $(E^*; m(E^*,E), w(E^*,E))$ are examples of locally convex linear bitopological spaces.[3]

1.3 COROLLARY. *Let $(E;\mathscr{T}_1,\mathscr{T}_2)$ be a locally convex linear bitopological space such that $(E,\mathscr{T}_1)$ is pseudo-metrizable and separable and $(E,\mathscr{T}_2)$ is Hausdorff. If K is a $\mathscr{T}_2$-compact subset of E, then the set of all points of continuity of the identity map: $(K,\mathscr{T}_2) \to (K,\mathscr{T}_1)$ is a dense G_δ subset of $(K,\mathscr{T}_2)$.*

Proof. Let $\{p_i\}$ be a sequence of $\mathscr{T}_1$-continuous, lower $\mathscr{T}_2$-semicontinuous pseudo-norms which determines $\mathscr{T}_1$. Let Z_i be the set of all points of continuity of the identity map: $(K,\mathscr{T}_2) \to (K,\mathscr{T}_{p_i})$. By Proposition 1.2, each Z_i is a dense G_δ subset of $(K,\mathscr{T}_2)$, and therefore $Z = \bigcap \{Z_i: i = 1,2,\cdots\}$ is also a dense G_δ subset of $(K,\mathscr{T}_2)$. The set Z is precisely the set of all points of continuity of the identity map $(K,\mathscr{T}_2) \to (K,\mathscr{T}_1)$.

In Proposition 1.2 and Corollary 1.3, the assumption of separability is necessary. The following example was suggested by E. Michael. Let M be a compact Hausdorff space such that no singleton is G_δ, let E be the Banach space $C(M)$ with the supremum norm, and let $e: M \to E^*$ be the evaluation map. Then $e[M]$ is a weak* compact subset of E^*. If the identity map: $(e[M], \text{weak*}) \to (e[M], \text{norm})$ is continuous at $e(m)$, $m \in M$, then $\{m\}$ is G_δ in M. Therefore this map is not continuous at any point of $e[M]$. For M, we may take for instance $[0,1]^{\aleph_1}$.

2. **The main results.** In Corollary 1.3, suppose K is convex as well as $\mathscr{T}_2$-compact. Then we may ask, as we have done in the introduction, whether the identity map $(K,\mathscr{T}_2) \to (K,\mathscr{T}_1)$ is continuous at any of the extreme points of K. We shall show below that this is indeed the case. We need the following theorem of Choquet. The symbol $\text{ext}(K)$ denotes the set of all extreme points of K. For a proof of Choquet's theorem, see [2; p. 355].

2.1 THEOREM. (Choquet). *Let $(E,\mathscr{T})$ be a Hausdorff locally convex linear*

(3) $s(E^*, E)$, $w(E^*, E)$ and $m(E^*, E)$ are respectively the strong, weak and Mackey topologies induced on E^* by the natural pairing of E^* and E.

topological space, and let K be a non-empty, compact, convex subset of E. Then $(\mathrm{ext}(K),\mathscr{T})$ *is of the 2nd category in itself.*

The essential idea of the proof of the next theorem is in the proof of the lemma in [6]. We shall, however, give the proof in full for the sake of completeness.

2.2 THEOREM. *Let* $(E,\mathscr{T})$ *be a Hausdorff locally convex linear topological space, let p be a lower* $\mathscr{T}$*-semicontinuous pseudo-norm on E such that* $(E,\mathscr{T}_p)$ *is separable, let K be a* $\mathscr{T}$*-compact, convex subset of E, and let Z be the set of all points of continuity of the identity map* $(K,\mathscr{T})\to(K,\mathscr{T}_p)$*. Then* $Z\cap\mathrm{ext}(K)$ *is a dense* G_δ *subset of* $(\mathrm{ext}(K),\mathscr{T})$.

Proof. We may assume that $K\neq\emptyset$. Let $X=\mathrm{ext}(K)$ and $\varepsilon>0$. Let B_ε be the subset of X such that $u\in B_\varepsilon$ if and only if there is a neighborhood of u in $(K,\mathscr{T})$ of p-diam $\leqq\varepsilon$. Clearly B_ε is an open subset of $(X,\mathscr{T})$. We will show that B_ε is dense in $(X,\mathscr{T})$.

Let W be an arbitrary $\mathscr{T}$-open subset of E such that $W\cap X\neq\emptyset$; we must show that $B_\varepsilon\cap W\neq\emptyset$. Let D be the $\mathscr{T}$-closure of $X=\mathrm{ext}(K)$. Then D is $\mathscr{T}$-compact and $W\cap D\neq\emptyset$. By prop. 1.2, the set of all points of continuity of the identity map: $(D,\mathscr{T})\to(D,\mathscr{T}_p)$ is dense in $(D,\mathscr{T})$. Hence there is a $\mathscr{T}$-open subset V of E such that $W\cap D\supset V\cap D\neq\emptyset$ and $p\text{-diam}(V\cap D)\leqq\varepsilon/2$. Let K_1 be the $\mathscr{T}$-closed, convex hull of the $\mathscr{T}$-compact set $D\sim V$, and let K_2 be the $\mathscr{T}$-closed convex hull of $D\cap V$. Since K_1 and K_2 are $\mathscr{T}$-compact and $X\subset K_1\cup K_2$, K is the convex hull of $K_1\cup K_2$. Note also that $p\text{-diam}(K_2)\leqq\varepsilon/2$, because p is lower $\mathscr{T}$-semicontinuous. Moreover $K_1\neq K$, because $\mathrm{ext}(K_1)\subset D\sim V$ and $D\cap V\neq\emptyset$. Let $r\in(0,1]$, and let C_r be the image of the map $f_r\colon K_1\times K_2\times[r,\ 1]\to K$ defined by $f_r(x_1,x_2,\lambda)=\lambda x_1+(1-\lambda)x_2$. Then C_r is a $\mathscr{T}$-compact, convex subset of K. In addition, $C_r\neq K$, because $X\cap C_r\subset K_1$ and $K_1\neq K$. Let $y\in K\sim C_r$. Then y is of the form $y=\lambda x_1+(1-\lambda)x_2$, $x_i\in K_i$, $\lambda\in[0,r)$. Hence $p(y-x_2)=\lambda p(x_1-x_2)\leqq rd$, where $d=p\text{-diam}(K)$. Now by the absorption theorem [3; p. 91], the set $\{x\colon p(x)\leqq 1\}$ absorbs K; hence K is $\mathscr{T}_p$-bounded, and it follows that $d<\infty$. Taking into account that $p\text{-diam}(K_2)\leqq\varepsilon/2$, we see that p-diam $(K\sim C_r)\leqq\varepsilon/2+2rd$. Let $C=C_r$ with $r=\varepsilon/4d$; then $p\text{-diam}(K\sim C)\leqq\varepsilon$. Since $C\neq K$, there is u in $(K\sim C)\cap X$, and $K\sim C$ is a neighborhood of u in $(K,\mathscr{T})$ of p-diam $\leqq\varepsilon$. Hence $u\in B_\varepsilon$. Next, since $C\supset K_1\supset D\sim V$, $u\in D\cap V\subset W$. Therefore $u\in B_\varepsilon\cap W$, and consequently $B_\varepsilon\cap W\neq\emptyset$. Hence B_ε is dense in $(X,\mathscr{T})$.

Finally, to conclude the proof, observe that $Z\cap X=\bigcap\{B_{1/n}\colon n=1,2,\cdots\}$ and that the set $\bigcap\{B_{1/n}\colon n=1,2,\cdots\}$ is a dense G_δ subset of $(X,\mathscr{T})$ in view of Theorem 2.1.

2.3 THE MAIN THEOREM. *Let* $(E;\mathscr{T}_1,\mathscr{T}_2)$ *be a locally convex linear bitopological space such that* $(E,\mathscr{T}_1)$ *is pseudo-metrizable and separable and* $(E,\mathscr{T}_2)$ *is Hausdorff, let K be a* $\mathscr{T}_2$*-compact, convex subset of E, and let Z be the set of* *all points of continuity of the identity map*: $(K,\mathcal{T}_2)\to(K,\mathcal{T}_1)$. *Then* $Z\cap\text{ext}(K)$ *is a dense* G_δ *subset of* $(\text{ext}(K),\mathcal{T}_2)$, *and the* $\mathcal{T}_2$*-closed, convex hull of* $Z\cap\text{ext}(K)$ *is* K.

Proof. Let Z_i be as in the proof of Corollary 1.3. Then, by Theorem 2.2, $Z_i\cap\text{ext}(K)$ is a dense G_δ subset of $(\text{ext}(K),\mathcal{T}_2)$. By Theorem 2.1, $(\text{ext}(K),\mathcal{T}_2)$ is of the 2nd category in itself. Therefore $\bigcap\{Z_i\cap\text{ext}(K)\colon i=1,2,\cdots\}\doteq[\bigcap\{Z_i\colon i=1,2,\cdots\}]\cap\text{ext}(K)=Z\cap\text{ext}(K)$ is a dense G_δ subset of $(\text{ext}(K),\mathcal{T}_2)$. The second conclusion follows from the first one, because of the Krein-Milman theorem.

§3. **Applications.** In this section, we shall derive several consequences of the material presented in §2. If $(E,\mathcal{T})$ is a locally convex, separable, metrizable, linear topological space, then the bitopological space $(E;\mathcal{T},w(E,E^*))$ satisfies the conditions of Theorem 2.3. Thus we obtain:

3.1 THEOREM. *Let* $(E,\mathcal{T})$ *be a locally convex, separable, metrizable, linear topological space, let* K *be a weakly compact* (i.e. $w(E,E^*)$*-compact*), *convex subset of* E, *and let* Z *be the set of all points of continuity of the identity map* $(K,w(E,E^*))\to(K,\mathcal{T})$. *Then* $Z\cap\text{ext}(K)$ *is weakly dense in* $\text{ext}(K)$, *and the closed convex hull of* $Z\cap\text{ext}(K)$ *is* K.

REMARKS (a). In Theorem 3.1, instead of the separability of $(E,\mathcal{T})$, one can assume that K is $\mathcal{T}$-separable or, equivalently, that K is weakly separable, because then the closed subspace generated by K will be separable.

(b) If E is a separable Banach space, then Theorem 3.1 is a consequence of Lindenstrauss' theorem [4; Theorem 4]. For, if u is a "strongly exposed" point of K, then the identity map $(K,\text{weak})\to(K,\text{norm})$ is continuous at u.

Let E be a Banach space such that E^* is separable. Then the bitopological space $(E^*;\mathcal{T},w(E^*,E))$ also satisfies the conditions of Theorem 2.3, where $\mathcal{T}$ is the norm topology for E^*. Thus we obtain:

3.2 THEOREM. *Let* E *be a Banach space such that* E^* *is separable, let* K *be a weak* compact, convex subset of* E^*, *and let* Z *be the set of all points of continuity of the identity map*: $(K,\text{weak}^*)\to(K,\text{norm})$. *Then* $Z\cap\text{ext}(K)$ *is weak* dense in* $\text{ext}(K)$.

3.3. COROLLARY. *Let* E *be a Banach space such that* E^* *is separable, let* K *be a norm-closed, bounded, convex subset of* E^*, *and let* K_1 *be the weak*-closure of* K. *Then* $K\cap\text{ext}(K_1)$, *which is a subset of* $\text{ext}(K)$, *is weak* dense in* $\text{ext}(K_1)$.

Proof. Since K is bounded, K_1 is weak* compact. Hence Theorem 3.2 applies to K_1. Let Z be the set of all points of continuity of the identity map $(K_1,\text{weak}^*)\to(K_1,\text{norm})$, and let $z\in Z$. Since K is weak* dense in K_1, there is a net $\{x_\alpha\}$ in K converging to z relative to the weak* topology. Then $\{x_\alpha\}$ converges to z relative to the norm topology, because $z \in Z$, and therefore $z \in K$. Hence $Z \subset K$ and $Z \cap \mathrm{ext}(K_1) \subset K \cap \mathrm{ext}(K_1)$. It follows from Theorem 3.2 that $K \cap \mathrm{ext}(K_1)$ is weak* dense in $\mathrm{ext}(K_1)$. Finally, since $K_1 \supset K$, $K \cap \mathrm{ext}(K_1) \subset \mathrm{ext}(K)$.

From Corollary 3.3, we may deduce a recent result due to Bessaga and Pełczyński [1]:

3.4 Corollary. *Let E be a Banach space such that E^* is separable. Then each norm-closed, convex, bounded subset of E^* is the norm-closed, convex hull of its extreme points.*

Proof. According to Lemma 1 in [5], it is sufficient to prove that, if K is a non-empty, norm-closed, convex bounded subset of E^*, then $\mathrm{ext}(K) \neq \varnothing$. Let K_1 be the weak* closure of K. Then $\mathrm{ext}(K_1) \neq \varnothing$, and hence, by Corollary 3.3, $\varnothing \neq K \cap \mathrm{ext}(K_1) \subset \mathrm{ext}(K)$.

Remark. As pointed out by Bessaga and Pełczyński [1], the separability of E^* is essential in Corollary 3.4.

A subset A of a Banach space E is called *dentable* [7], if, for each $\varepsilon > 0$, there is x in A such that x is *not* in the convex, closed hull of $A \sim \{y: \|x - y\| \leqq \varepsilon\}$. It is easily seen that A is dentable if the closed, convex hull of A is. A point x of A is called a *denting point* if for each $\varepsilon > 0$, x does not belong to the closed, convex hull of $A \sim \{y: \|x - y\| \leqq \varepsilon\}$. Clearly A is dentable if the closed, convex hull of A contains a denting point. A denting point is necessarily extreme. It is easy to deduce from Theorem 3.1 that a relatively weakly compact subset of a separable Banach space is dentable. M. Rieffel asked [7; Question 1] whether the separability here is essential. This question, we believe, is still open. Rieffel also raised the question [7; Question 3]: which Banach spaces have the property that all bounded subsets are dentable? He proved that $l^1(X)$ has this property, where X is any set (possibly uncountable). We give another family of Banach spaces with this property.

3.5 Theorem. *Let E be a Banach space such that E^* is separable. Then each nonempty, norm-closed, convex, bounded subset of E^* contains a denting point. Consequently each bounded subset of E^* is dentable.*

Proof. Let K be a nonempty, norm-closed, convex, bounded subset of E^* let K_1 be its weak* closure. Let u be a point in $\mathrm{ext}(K_1)$ where the identity map: $(K_1, \text{weak*}) \to (K_1, \text{norm})$ is continuous. By Theorem 3.2, we know such a point exists, and, as shown in the proof of Corollary 3.3, $u \in \mathrm{ext}(K)$. Let $\varepsilon > 0$. Then there is a weak* open subset W of E^* such that $u \in W \cap K_1$ and $\mathrm{diam}(W \cap K_1) \leqq \varepsilon$. The point u is not in the weak* closed, convex hull of $K_1 \sim W$, and *a fortiori* u is not in the (norm)-closed, convex hull of $K_1 \sim W$. Clearly $K_1 \sim W \supset K \sim \{x: \|x - u\| \leqq \varepsilon\}$, and hence u is not in the closed, convex hull of $K \sim \{x: \|x-u\| \leqq \varepsilon\}$. Therefore u is a denting point of K. The second conclusion is clear from the remarks preceding the theorem.

Let E be a locally convex, metrizable linear topological space, let E^* be its dual with the strong topology (i.e. $s(E^*,E)$), and let E^{**} be the dual of E^* with the strong topology (i.e. $s(E^{**},E^*)$). Let I denote the evaluation map: $E \to E^{**}$. Then I is one-to-one and relatively open. Since E is metrizable, the topology is bound [3; 22.3], and hence it is evaluable [3; 20.4], i.e. I is continuous. Therefore I is a linear topological isomorphism of E onto $I[E]$. A *Fréchet space* is a locally convex, complete metrizable linear topological space. A linear topological space is called *quasi-separable*, if each bounded subset is separable.

3.6. THEOREM. *Let E be a Fréchet space such that E^{**} is quasi-separable relative to the strong topology. Then each closed, bounded, convex subset of E is the closed convex hull of its extreme points.*

Proof. Let K be a closed, bounded, convex subset of E, abd let I be the evaluation map: $E \to E^{**}$. Then the weak* ($= w(E^{**},E^*)$) closure K_1 of $I[K]$ in E^{**} is weak* compact and strongly bounded. Let F be the subspace of E^{**} generated by K_1. Then, since E^{**} is quasi-separable, $(F, s(E^{**},E^*))$ is separable and metrizable, and thus we may apply Theorem 2.3 to the bitopological linear space $(F; s(E^{**},E^*), w(E^{**},E^*))$ and K_1. Let Z be the set of all points of continuity of the identity map: $(K_1, \text{weak*}) \to (K_1, \text{strong})$. Because K is complete, $I[K]$ is strongly closed in E^{**}, and, as in the proof of Corollary 3.3, we see that $Z \subset I[K]$. Hence $Z \cap \text{ext}(K_1) \subset \text{ext}(I[K]) = I[\text{ext}(K)]$, and Theorem 2.3 implies that the weak* closed, convex hull of $I[\text{ext}(K)]$ is K_1. It follows that the (weak) closed, convex hull of $\text{ext}(K)$ is K.

REMARK. Let E be a Fréchet space satisfying the hypothesis of Theorem 3.6. Then, by [3; 22.15], each strongly bounded subset of E^{**} is equicontinuous, i.e. E^* is evaluable. Therefore, by [3; 22.15], E^* is bound and barrelled as well; that is, E is a distinguished ($=$ *distingué*) Fréchet space.

Finally, we present a bitopological version of Ryll-Nardzewski's fixed point theorem. Let Q be a subset of a locally convex linear topological space $(E, \mathscr{T})$ and let $\mathscr{S}$ be a semi-group of transformations of Q into Q. The semigroup $\mathscr{S}$ is *$\mathscr{T}$-noncontracting on Q*, if, for each pair of distinct points x, y of Q, *there* is a $\mathscr{T}$-continuous pseudo-norm p on E such that:

$$\inf\{p(Tx - Ty): T \in \mathscr{S}\} > 0.$$

A proof of the next theorem is a straightforward modification of the proof of Ryll-Nardzewski's fixed point theorem given in [6]. Theorem 2.2 takes the place of the lemma in [6].

3.7. THEOREM. *Let $(E; \mathscr{T}_1, \mathscr{T}_2)$ be a locally convex linear bitopological space such that $(E, \mathscr{T}_1)$ is separable and $(E, \mathscr{T}_2)$ is Hausdorff, let Q be a nonempty,*

$\mathscr{T}_2$-compact, convex subset of E, and let $\mathscr{S}$ be a semigroup of $\mathscr{T}_2$-continuous affine transformations on Q into itself. If $\mathscr{S}$ is $\mathscr{T}_1$-noncontracting on Q then $\mathscr{S}$ has a common fixed point in Q.

REMARK. In Theorem 3.7, the assumption of the separability of $(E,\mathscr{T}_1)$ can be dropped if $(E;\mathscr{T}_1,\mathscr{T}_2)$ satisfies the following condition:

(S) Each $\mathscr{T}_2$-compact, $\mathscr{T}_2$-separable subset of E is included in a $\mathscr{T}_1$-separable subset of E.

If $(E,\mathscr{T})$ is a locally convex space then $(E;\mathscr{T}, \text{weak})$ satisfies (S). However, if E is a separable Banach space, then $(E^*;\text{norm},\text{weak}^*)$ satisfies (S) if and only if E^* is separable relative to the norm topology.

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